TECHNICAL NOTE NO. 11

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# GENERATION OF NONSTATIONARY RANDOM PROCESSES DEPENDENT ON TIME AND POSITION PARAMETERS

By David L. Finn

Prepared for
George C. Marshall Space Flight Center
Huntsville, Alabama

Contract No. NAS8-2473

(Development of New Methods and Applications of Analog Computation)

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# ABSTRACT

Two methods are presented for the synthesis of analog computer networks that approximate a random process depending on both time and a position parameter. The networks are intended for use in the simulation of random wind disturbances that affect a rocket or other aerospace vehicle in flight. The output of the analog computer network simulates the effect of the prescribed random process on the vehicle as its position varies arbitrarily with time. The two inputs to the analog computer network are a Gaussian white-noise random process and a function of time characterizing the variable position of the vehicle.

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## I. INTRODUCTION

In this technical note two methods are presented for the synthesis of analog computer networks that approximate a random process depending on both time and a position parameter. The networks are intended for use in the simulation of random wind disturbances that affect a rocket or other aerospace vehicle in flight. Each network produced by either of the synthesis procedures has two inputs. One input is a Gaussian white-noise process and the second input is a time function characterizing the position of the moving vehicle. The parameter characterizing position may be taken to be altitude in the wind disturbance application. The output of each network is a time-parameter Gaussian random process that approximates the instantaneous random disturbance affecting a rocket or other sensing element as it undergoes an arbitrary position variation in time. It is not necessary that the position variation of the sensing element be specified a priori. Thus, in the wind simulation application it is necessary to synthesize only a single network to represent the disturbances affecting vehicles having a wide variety of flight trajectories.

It is assumed that the first statistical moment of the time/position-parameter process being approximated is identically zero and that the second statistical moment is known. Processes having a nonzero first statistical moment may be generated by the addition of a deterministic function to the random process generated by one of the procedures described in this technical note. The output of the analog computer network derived by each of the procedures is a time-parameter Gaussian random process having first and second statistical moments approximating the moments of the original time/position-parameter random process.

The two synthesis procedures will be designated in this technical note as the "covariance-expansion method" and as the "spectral-density method." The covariance-expansion method is a direct adaptation of the synthesis method for the realization of time-parameter random processes that was presented in Technical Note No. 3 on this project (Reference 8) and further developed in the thesis of Reference 9. The spectral-density method is a direct adaptation of the synthesis method for the realization of position-parameter random processes that was presented in Technical Note No. 10 (Reference 3).

One method of obtaining experimental wind data utilizes a tracking radar to record the effects of wind disturbance on the motion of an ascending Jimsphere balloon (Reference 7). This data may be used for the calculation of a covariance function that depends on altitude alone. Data in this form is best suited for use with the covariance-expansion synthesis procedure discussed in Chapter II of this technical note. The mechanization system derived by use of data in this form is best suited to simulate the wind disturbances that affect a vehicle in nearly vertical flight. However, the synthesis procedure of Chapter II is not at all restricted for use in vertical flight patterns. It is of greater generality than the method of Chapter III. At the same time, unfortunately, it is of greater mathematical complexity, and in general requires a greater amount of statistical data for implementation.

A second method of obtaining experimental wind data involves the recording of wind effects on a sensor maintained at fixed altitude (Reference 6). This data may be used to calculate a power spectral density function if it is assumed that the wind disturbance at a fixed altitude may be represented as a stationary random process. Data expressed in this form, for a succession of altitudes, is best suited for use with the spectral-density synthesis procedure discussed in Chapter III of this technical note. The mechanization system derived by use of data in this form is best suited to simulate the wind disturbances that affect a vehicle in nearly horizontal flight. It is to be noted that even though the wind disturbance is assumed to be a stationary random process depending on time for each fixed altitude, the process may be nonstationary in the altitude parameter.

Unlike the procedure of Chapter III, the covariance-expansion synthesis method of Chapter II makes no assumption of stationarity. The random process to be approximated may be nonstationary in both time and position parameters.

Chapter II of this technical note is devoted to a presentation of the covariance-expansion synthesis procedure. A simple example is worked to clarify the procedure. Chapter III is devoted to a presentation of the spectral-density synthesis procedure. An example is also worked in that chapter to demonstrate application of the procedure. Chapter IV contains a discussion of conclusions derived on the basis of the research work done to date and contains some comments concerning future plans for the research.

## II. THE COVARIANCE-EXPANSION SYNTHESIS PROCEDURE

### 2-1. Introduction

This chapter is devoted to a presentation of the covariance-expansion synthesis procedure. The output of the analog computer network derived by this procedure approximates a random process $g(z,t)$ depending on position $z$ and time $t$. Specifically, the output of the network approximates the composite random process $x(t) = g(z(t),t)$. The process $x(t)$ represents the effect of the original time/position random process on a sensing element having the position $z$ at the time instant $t$. The representation is exact provided the random process is Gaussian, has a covariance expansion of the form of (2.2), and provided the position variable $z$ is a monotone function of time. Otherwise, the output of the analog computer network is an approximation to the original random process.

The notation used in this technical note has been chosen to conform to that used in Technical Note No. 3 (Reference 8). The synthesis procedure presented in this chapter is a direct adaptation of the procedure in Technical Note No. 3. In order to save space, those details of proof of validity of the procedure that appear in Technical Note No. 3 are not repeated here. Only those steps are included that are necessary to demonstrate implementation of the synthesis method.

### 2-2. The Synthesis Procedure

The time/position-parameter random process to be approximated is denoted as $g(z,t)$. Here, $z$ is a variable denoting the instantaneous position at time $t$ of a sensing element that is affected by the random process.

The covariance function for the random process $g(z,t)$ is denoted as

$$r_g(z',t',z,t) = E\,[g(z_1,t_1)g(z_2,t_2)] \tag{2.1}$$

where E is the expectation operation and

$$z' = \text{larger of } (z_1 \text{ and } z_2)$$

$$z = \text{smaller of } (z_1 \text{ and } z_2)$$

$$t' = \text{larger of } (t_1 \text{ and } t_2)$$

$$t = \text{smaller of } (t_1 \text{ and } t_2)$$

Here it is assumed that the expected value of $g(z,t)$ is identically zero. Processes with nonzero mean may be generated by the addition of $g(z,t)$ to a deterministic function.

It is assumed that the covariance function of the random process $g(z,t)$ may be expressed as a finite series expansion in the form

$$r_g(z', t', z, t) = \sum_{i=1}^{n} \eta_i(z', t') \quad \theta_i(z, t) \tag{2.2}$$

where $\eta_i(z', t')$ and $\theta_i(z, t)$ are known functions of time and position. If equation (2.2) does not apply exactly, the covariance function $r_g(z', t', z, t)$ must be approximated by an expansion of this form.

It is recalled that the position variable z denotes the instantaneous position at time t of a sensing element that is affected by the random process $g(z, t)$. Throughout this technical note it will be assumed that the position of the sensing element is described by a nondecreasing function of time $z(t)$. However, the synthesis procedure developed is equally valid if the position is described by a nonincreasing function of time.

An analog computer network is to be synthesized having two inputs--a Gaussian white-noise waveform, and the function $z(t)$ representing the instantaneous position of a sensing element. The output of the network is to be a composite random process

$$x(t) = g(z(t), t) \tag{2.3}$$

The composite time-parameter random process $x(t)$ represents the effect of the time/position-parameter random process $g(z,t)$ on the sensing element having the instantaneous position $z(t)$.

The covariance function of the random process $x(t)$ is denoted as $r(t', t)$. By use of (2.3) and (2.1) this may be expressed as

$$r(t', t) = E[x(t_1)x(t_2)] = r_g(z(t'), t', z(t), t) \tag{2.4}$$

By use of (2.2) the covariance function $r(t', t)$ may be represented as a finite series expansion

$$r(t', t) = \sum_{i=1}^{n} \phi_i(t') \gamma_i(t) \tag{2.5}$$

where

$$\phi_i(t') = \eta_i(z(t'), t')$$

$$\gamma_i(t) = \theta_i(z(t), t)$$

The analog computer network to be synthesized is characterized by the nth order differential equation

$$x^{(n)} + p_{n-1}(t)\, x^{(n-1)} + \ldots + p_1(t)\, x^{(1)} + p_o(t)\, x$$

$$= q_{n-1}(t)\, y^{(n-1)} + \ldots + q_1(t) y^{(1)} + q_o(t) y \tag{2.6}$$

Here $x^{(k)}$ denotes the kth derivative of the function x with respect to time. In shorter notation, (2.6) may be written as

$$L_t\, x = N_t\, y \tag{2.7}$$

The function y represents the Gaussian white-noise input to the analog computer network.

In order to avoid differentiation of the noise input y, the equation of (2.7) may be converted into a set of n first-order differential equations. To make this conversion the following identifications are utilized:

$$x(t) = x_1(t) \tag{2.8}$$

$$x_1^{(1)} = x_2 - a_{n-1}(t)\, x_1 + b_{n-1}(t)\, y$$

$$x_2^{(1)} = x_3 - a_{n-2}(t)\ x_1 + b_{n-2}(t)\ y$$

$$x_{n-1}^{(1)} = x_n - a_1(t)\ x_1 + b_1(t)\ y$$

$$x_n^{(1)} = -\ a_o(t)\ x_1 + b_o(t)\ y$$

This set can be written more concisely in matrix notation as

$$\underline{x}^{(1)} = A(t)\ \underline{x} + B(t)\ y \tag{2.9}$$

$$[x] = [x_1] = H\ \underline{x}$$

where

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} \qquad B(t) = \begin{bmatrix} b_{n-1}(t) \\ b_{n-2}(t) \\ \cdot \\ \cdot \\ b_1(t) \\ b_o(t) \end{bmatrix}$$

$$H = [\,1\ \ 0\ \ \cdot\ \ \cdot\ \ 0\ \ 0\,]$$

$$A(t) = \begin{bmatrix} -\ a_{n-1}(t) & 1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ -\ a_{n-2}(t) & 0 & 1 & 0 & \cdot & \cdot & 0 & 0 \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ -\ a_1(t) & 0 & 0 & 0 & \cdot & \cdot & 0 & 1 \\ -\ a_o(t) & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \end{bmatrix}$$

The elements $a_k$, $b_k$ in (2.8) are related to the coefficients $p_k$, $q_k$ in (2.6) by

$$p_k = \sum_{j=0}^{n-1-k} \frac{(n-1-j)!}{k!\,(n-1-j-k)!} \qquad a_{n-1-j}^{(n-1-j-k)} \tag{2.10}$$

$$q_k = \sum_{j=0}^{n-1-k} \frac{(n-1-j)!}{k!\,(n-1-j-k)!} \qquad b_{n-1-j}^{(n-1-j-k)} \tag{2.11}$$

If the $p_k$, $q_k$ are shown, then (2.10) and (2.11) can be solved sequentially for the $a_k$, $b_k$.

Associated with the vector differential equation of (2.9) is the homogeneous equation

$$\underline{x}^{(1)} = A(t)\ \underline{x} \tag{2.12}$$

The $\phi_1(t)$, $\phi_2(t)$, $\cdot\ \cdot$ $\phi_n(t)$ of (2.5), which are taken to be linearly independent, may be used in the construction of a fundamental matrix solution $\Phi(t)$ satisfying

$$\frac{d}{dt}\ \Phi(t) = A(t)\ \Phi(t) \tag{2.13}$$

The matrix $\Phi(t)$ is defined as

$$\Phi(t) = \begin{bmatrix} \phi_{11} & \phi_{21} & \cdot\ \cdot & \phi_{n1} \\ \phi_{12} & \phi_{22} & \cdot\ \cdot & \phi_{n2} \\ \cdot & & & \\ \cdot & & & \\ \phi_{1n} & \phi_{2n} & \cdot\ \cdot & \phi_{nn} \end{bmatrix} \tag{2.14}$$

where

$$\phi_{ij} = \frac{d}{dt}\,\phi_{ij-1} + a_{n-j+1}\,\phi_{i1} \tag{2.15}$$

and

$$\phi_{k1} = \phi_k(t) \qquad \{k = 1, 2, \cdot \cdot n\}$$

$$= \alpha_k(z(t), t)$$

It is noted that the chain rule is to be used in the differentiation of $\phi_k(t)$.

$$\frac{d}{dt}\,\phi_k(t) = \frac{d}{dt}\,\alpha_k(z(t), t) \tag{2.16}$$
$$= \frac{\partial\alpha_k}{\partial t} + \frac{\partial\alpha_k}{\partial z}\;\frac{dz}{dt}$$

The nonhomogeneous differential equation of (2.9) with zero initial conditions has the unique solution

$$\underline{x}(t) = \int_0^t \Phi(t)\;\Phi^{-1}(s)\;B(s)\;y(s)\;ds \tag{2.17}$$

where $\Phi^{-1}(s)$ is the matrix inverse of $\Phi(t)$.

A covariance matrix for the vector $\underline{x}(t)$ can be written as

$$R(t', t) = E\left[\;\underline{x}(t_1)\;\underline{x}^T(t_2)\right] \tag{2.18}$$

$$= \int_0^t \Phi(t')\;\Phi^{-1}(s)\;B(s)\;B^T(s)\left[\Phi^{-1}(s)\right]^T\;\Phi^T(t)\;ds$$

where superscript T denotes matrix transpose.

The coefficients $a_k$, $b_k$ of (2.8) will now be determined. Associated with the differential equation of (2.7) is the homogeneous differential equation

$$L_t(x) = 0 \tag{2.19}$$

The linear differential operator $L_t$ can be specified in terms of the $\phi_k$ of (2.5) by the relation

$$L_t\, x = W(x, \phi_1, \phi_2, \cdot\cdot, \phi_n) = 0 \tag{2.20}$$

where the Wronskian W is given by

$$W(x, \phi_1, \phi_2, \cdot \cdot, \phi_n) = \begin{bmatrix} x & \phi_1 \cdots & \phi_n \\ x^{(1)} & \phi_1^{(1)} \cdot\cdot & \phi_n^{(1)} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ x^{(n)} & \phi_1^{(n)} \cdot\cdot & \phi_n^{(n)} \end{bmatrix} \tag{2.21}$$

The coefficients $p_k$ can be obtained using equation (2.20). The elements $a_k$ that appear in (2.8) and in the matrix A of (2.9) can be obtained directly by using (2.10).

The elements $b_k$ of (2.8) and (2.9) now must be determined. Once the $a_k$ are determined by the procedure described above, the matrix $\Phi(t)$ can be written using equation (2.15). The matrix covariance expression of (2.18) can be written in the form

$$R(t', t) = \Phi(t')\, D(t)\, \Phi^T(t) \tag{2.22}$$

where the elements $d_{ij}(t)$ of the matrix D(t) may be expressed in terms of the $\phi_i(t)$ and $\gamma_i(t)$ of (2.5) in the form

$$\begin{aligned} d_{ij}(t) &= \frac{\gamma_i(t)}{\phi_i(t)} && i = j \\ &= 0 && i \neq j \end{aligned} \tag{2.23}$$

The determination of the elements of the matrices $\Phi(t)$ and D(t) allows the determination of the elements of $R(t', t)$ by use of (2.22).

A matrix $R^*(t', t)$ is now defined as

$$R^*(t', t) = R^T(t, t') \tag{2.24}$$

$$= \Phi(t)\, D(t')\, \Phi^T(t')$$

Let $\Delta(t', t)$ be defined as

$$\Delta(t', t) = R(t', t) - R^*(t', t) \tag{2.25}$$

Finally, let $\delta_{ii}$ be the diagonal elements of the matrix

$$\left. \frac{\partial}{\partial t'} \Delta(t', t) \right|_{t' = t} \tag{2.26}$$

It can be shown that

$$b_{n-i} = \sqrt{-\delta_{ii}}, \quad i = 1, 2, \cdot \cdot, n \tag{2.27}$$

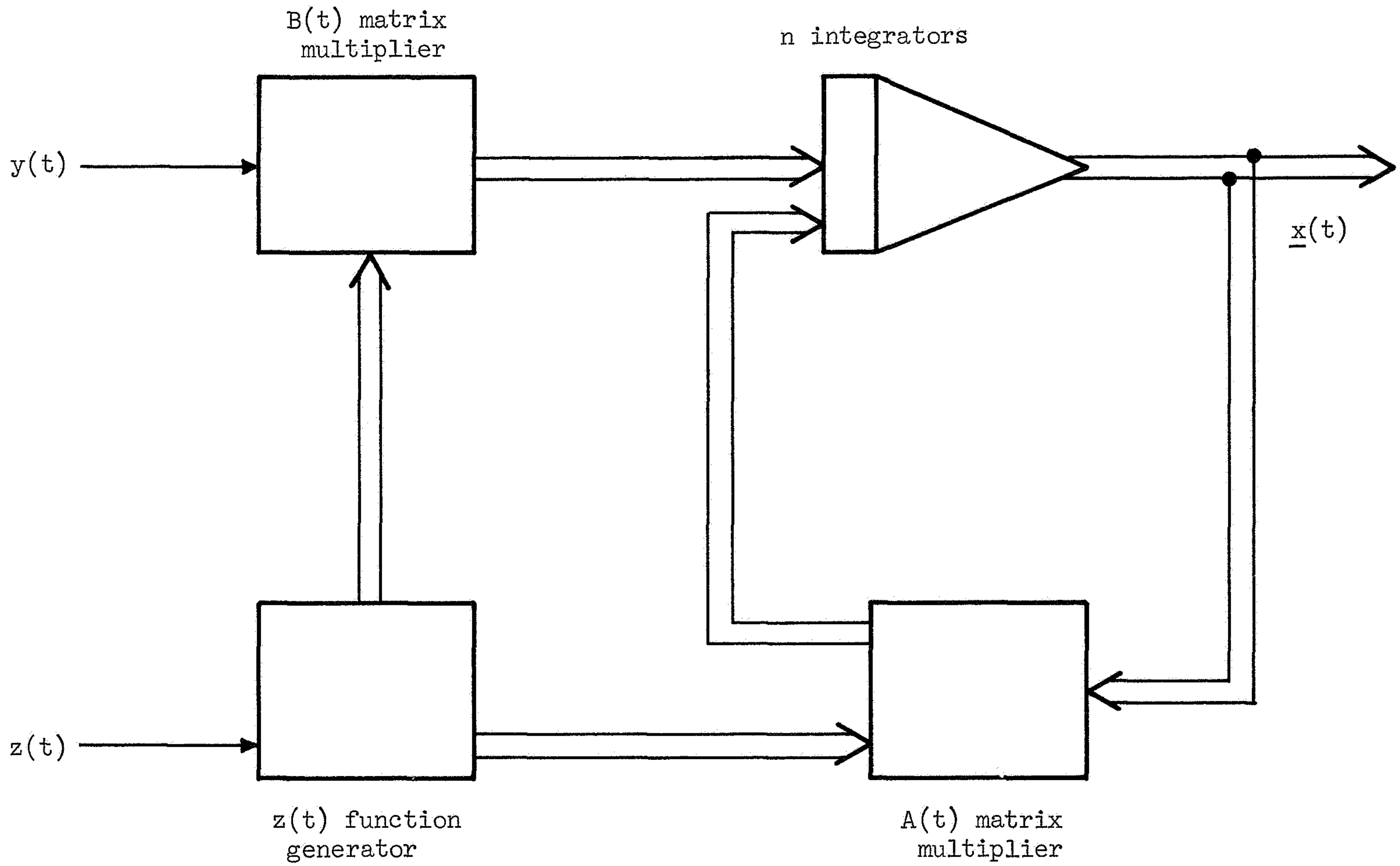
The matrix B(t) is now specified, thus completing the synthesis procedure. The mechanization system that generates the random process x(t) is characterized by the set of differential equations in (2.8). A block diagram of the mechanization system is shown in Figure 2.1.

It is important to achieve the proper integrator initial conditions at the beginning of a computation cycle ($t = 0$) if the output of the mechanization system is to realize the correct covariance function. Different initial conditions applied to the same mechanization system with the same position-function excitation may give rise to widely divergent covariance functions. In general, the initial conditions may be altered by manipulation of the white-noise input and the position-parameter z(t) input during resetting of the analog computer just prior to $t = 0$.

## 2-3. An Example

A numerical example is presented in this section to help clarify the covariance-expansion synthesis procedure.

The covariance function for the random process g(z, t) is assumed to be




Note: Double lines indicate multivariable signal flow.

Figure 2.1. Mechanization System Derived by the Covariance-Expansion Synthesis Procedure.

$$r_g(z', t', z, t) = c^2 e^{-\alpha z'} e^{-\beta(t' - t)} \tag{2.28}$$

The coefficients in (2.2) are seen from the above to be

$$\eta_1(z', t') = c\, e^{-\alpha z'} e^{-\beta t'} \tag{2.29}$$

$$\theta_1(z, t) = c\, e^{\beta t}$$

In a similar manner, the coefficients in (2.5) are seen to be

$$\phi_1(t') = c\, e^{-\alpha z(t') - \beta t'} \tag{2.30}$$

$$\gamma_1(t) = c\, e^{\beta t}$$

The $p_k$ coefficients of (2.6) are determined by use of the relation in (2.20):

$$L_t(x) = \begin{vmatrix} x & \phi_1(t) & \\ x^{(1)} & \phi_1^{(1)}(t) & = 0 \end{vmatrix} \tag{2.31}$$

Expanding this determinant with the use of (2.30), there results

$$x^{(1)} + \left(\beta + \alpha z^{(1)}(t)\right) x = 0 \tag{2.32}$$

A comparison of (2.32) and (2.6) shows that

$$p_o(t) = \beta + \alpha z^{(1)}(t) \tag{2.33}$$

Utilization of (2.10) provides the single coefficient of the A(t) matrix of (2.9):

$$a_o(t) = p_o(t)$$

$$= \beta + \alpha z^{(1)}(t) \tag{2.34}$$

The single coefficient of the $\Phi(t)$ matrix of (2.14) is found by use of (2.15) and (2.30):

$$\phi_{11} = \phi_1(t) = c\, e^{-\alpha z(t) - \beta t} \tag{2.35}$$

The single coefficient of the D(t) matrix of (2.22) is found by use of (2.23):

$$d_{11} = \frac{\gamma_1(t)}{\phi_1(t)} = e^{\alpha z(t) + 2\beta t} \tag{2.36}$$

The single element matrix $R(t', t)$ of (2.22) is found by use of (2.35) and (2.36):

$$R(t', t) = \Phi(t')\, D(t)\, \Phi^T(t) \tag{2.37}$$

$$= \left[ c^2\, e^{-\alpha z(t')}\, e^{-\beta t'}\, e^{\beta t} \right]$$

The matrix $R^*(t', t)$ of (2.24) is

$$R^*(t', t) = R^T(t, t') \tag{2.38}$$

$$= \left[ c^2\, e^{-\alpha z(t)}\, e^{-\beta t}\, e^{\beta t'} \right]$$

The difference matrix of (2.25) is

$$\Delta(t', t) = R(t', t) - R^*(t', t) \tag{2.39}$$

$$= \left[ c^2\, e^{-\alpha z(t')}\, e^{-\beta t}\, e^{\beta t'} - c^2\, e^{-\alpha z(t)}\, e^{-\beta t'}\, e^{\beta t} \right]$$

The matrix of (2.26) is found by use of (2.39):

$$\frac{\partial}{\partial t} \Delta(t', t) \Bigg|_{t'=t} = [\delta_{11}] \tag{2.40}$$

$$= \left[ - c^2 e^{-\alpha z(t)} \left( 2\beta + \alpha z^{(1)}(t) \right) \right]$$

Finally, the single coefficient in the B(t) matrix of (2.9) is found by use of (2.27) and (2.40):

$$b_o = \sqrt{-\delta_{11}}$$

$$= c\ e^{-\alpha z(t)/2} \sqrt{2\beta + \alpha z^{(1)}(t)}$$

Specification of this coefficient completes the synthesis procedure. The differential equation characterizing the mechanization system for the realization of the random process $x(t) = g(z(t), t)$ is obtained by substitution of the coefficients of (2.34) and (2.41) in the system of (2.8):

$$\frac{dx(t)}{dt} = - a_o(t)\ x(t) + b_o(t)\ y(t) \tag{2.42}$$

$$= - \left( \beta + \alpha \frac{dz(t)}{dt} \right) x(t) + c\ e^{-\alpha z(t)/2} \sqrt{2\beta + \alpha \frac{dz(t)}{dt}}\ \ y(t)$$

A block diagram of the mechanization system that implements the differential equation (2.42) is shown in Figure 2.2.

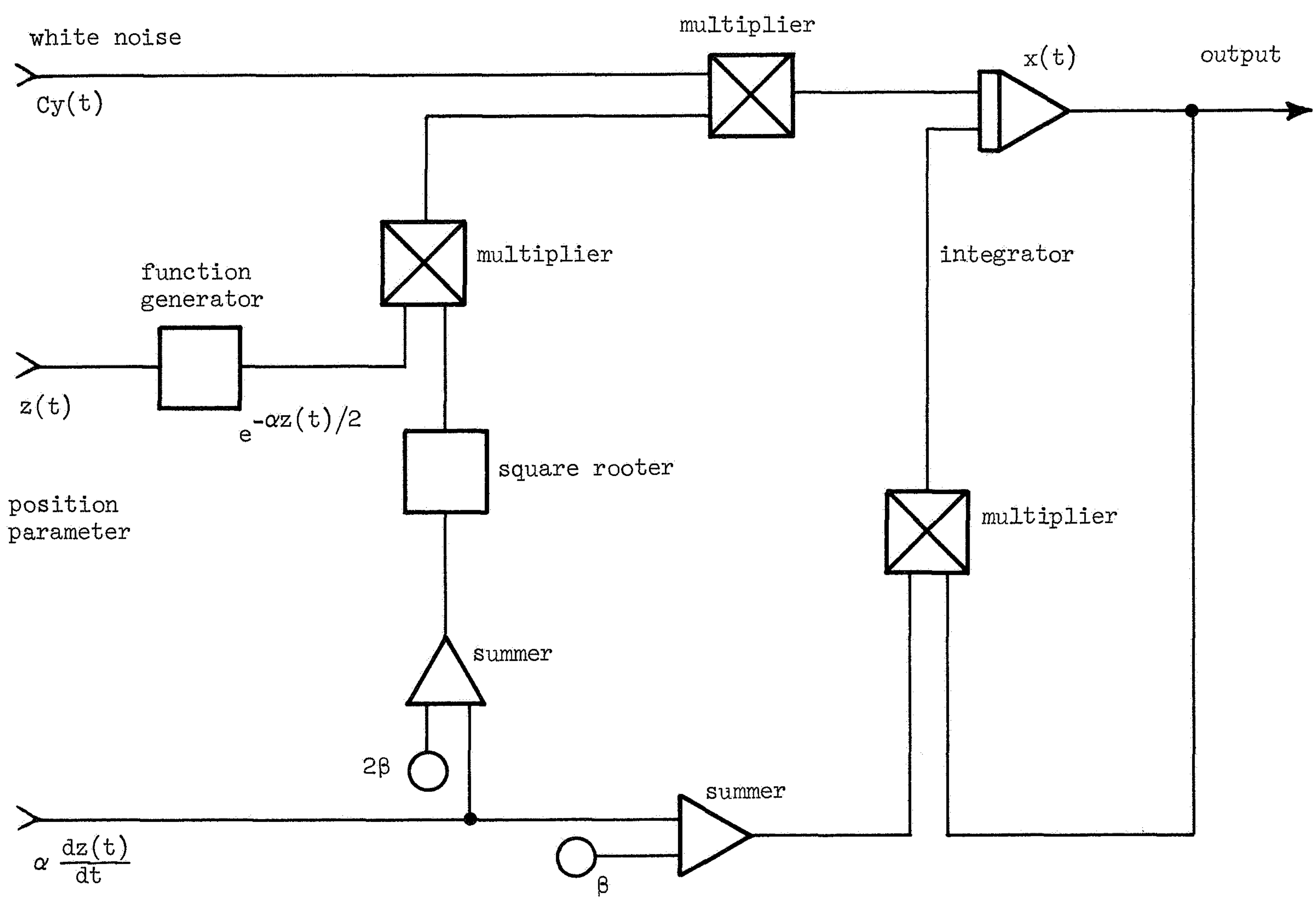



Figure 2.2. Mechanization System for the Realization of the Random Process of the Example in Section 2-3.

## III. THE SPECTRAL-DENSITY SYNTHESIS PROCEDURE

### 3-1. Introduction

This chapter is devoted to a presentation of the spectral-density synthesis procedure. The output of the analog computer network derived by this procedure approximates a random process $g(z, t)$ depending on position $z$ and time $t$. Specifically, the output of the network approximates the composite random process $x(t) = g(z(t), t)$. The process $x(t)$ represents the effect of the original time/position-parameter random process on a sensing element having the position $z$ at the instant $t$. The representation is exact for each fixed position $z$ provided the original random process is stationary in time for that position and has power spectral density in the form of equation (3.1) below. Otherwise, the output of the analog computer network is an approximation to the original process. The approximation is best when $z$ is a slowly varying function of time.

### 3-2. The Synthesis Procedure

The time/position-parameter random process to be approximated is denoted as $g(z,t)$. Here, $z$ is a variable denoting the instantaneous position at time $t$ of a sensing element that is affected by the random process.

It is assumed that the expected value of $g(z, t)$ is identically zero. Processes with nonzero mean may be generated by the addition of $g(z, t)$ to a deterministic function.

At any fixed position $z$, it is assumed that the random process $g(z, t)$ is wide-sense stationary in the time parameter $t$. The power spectral density at the position $z$ is denoted as $S_z(\omega)$. It is assumed that this power spectral density may be expressed as a ratio of polynomials in $\omega$.

$$S_z(\omega) = \frac{c_{2n-2}(z)\,\omega^{2n-2} + \cdots + c_2(z)\,\omega^2 + c_o(z)}{d_{2n}(z)\,\omega^{2n} + \cdots + d_2(z)\,\omega^2 + d_o(z)} \tag{3.1}$$

The c and d coefficients of $\omega^k$ are functions of position $z$. If equation (3.1) does not apply exactly, the power spectral density must be approximated by an expression of this form.

The power spectral density of (3.1) may be expressed as follows (see

Reference 1, page 227):

$$S_z(\omega) = \left| \, \left| H_z(s) \right|^2 \right|_{s = j\omega} \tag{3.2}$$

Here, $H_z(s)$ is the transfer function of a time-invariant linear filter. Physically, for any fixed position z the application of a white-noise random process to the input of a filter having transfer function $H_z(s)$ produces at the output a random process x(t) having power spectral density $S_z(\omega)$.

The transfer function $H_z(s)$ may be expressed as a ratio of two polynomials in complex frequency s.

$$H_z(s) = \frac{b_{n-1}(z)s^{n-1} + \cdot \cdot + b_1(z)s + b_o(z)}{s^n \quad + \cdot \cdot + a_1(z)s + a_o(z)} \tag{3.3}$$

The $a_k$ and $b_k$ coefficients of $s^k$ are functions of position z. A technique for determining these coefficients if $S_z(\omega)$ is known, is given on page 233 of Reference 1.

The transfer function $H_z(s)$ of (3.3) may be realized in a variety of ways by the use of analog computer components (see References 4 and 5).

In particular, this transfer function may be realized for any fixed z by a mechanization of the differential equation

$$\frac{d^n x}{dt^n} + \cdot \cdot + a_1(z) \frac{dx}{dt} + a_o(z) \, x \tag{3.4}$$

$$= b_{n-1}(z) \frac{d^{n-1}y}{dt^{n-1}} + \cdot \cdot + b_1(z) \frac{dy}{dt} + b_o(z) \, y$$

Here, y represents the white-noise random process that is applied as input to the mechanization system. The output of the mechanization system is the random process x(t) = g(z, t) for the particular fixed value of position z previously selected.

A method of mechanizing the transfer function of (3.3) for fixed position z that avoids differentiation of the white-noise input y may be obtained by converting the differential equation of (3.4) to a set of first-order differential equations equivalent to (3.4) for any fixed value of z (see Reference 8, page 8):

$$x_1 = x \tag{3.5}$$

$$\frac{dx_1}{dt} = x_2 - a_{n-1}(z)\ x_1 + b_{n-1}(z)\ y$$

$$\frac{dx_2}{dt} = x_3 - a_{n-2}(z)\ x_1 + b_{n-2}(z)\ y$$

$$\cdot \quad \cdot \quad \cdot$$

$$\frac{dx_{n-1}}{dt} = x_n - a_1(z)\ x_1 + b_1(z)\ y$$

$$\frac{dx_n}{dt} = -\ a_o(z)\ x_1 + b_o(z)\ y$$

As was discussed previously, a mechanization of the set of equations in (3.5) provides an exact realization of the random process $x(t) = g(z, t)$ for any fixed value of position z. It might therefore be expected that the mechanization of this set of equations for variable z provides an approximate realization of the composite random process $x(t) = g(z(t), t)$ when the parameter z is allowed to become a slowly-varying function of time z(t). An analog computer network that mechanizes the set of equations in (3.5) when the position parameter is described by a function of time z(t) is shown in Figure 3.1.

It is again important to achieve the proper integrator initial conditions at the beginning of a computation cycle ($t = 0$) if the output of the mechanization system is to realize the correct random process. This is relatively straightforward to accomplish with this mechanization system. The white-noise input y(t) is applied to the network at all times. During the reset operation of the analog computer just prior to $t = 0$, the position-parameter input z(t) is maintained constant at its desired initial value long enough for the output random process x(t) to become stationary in time.

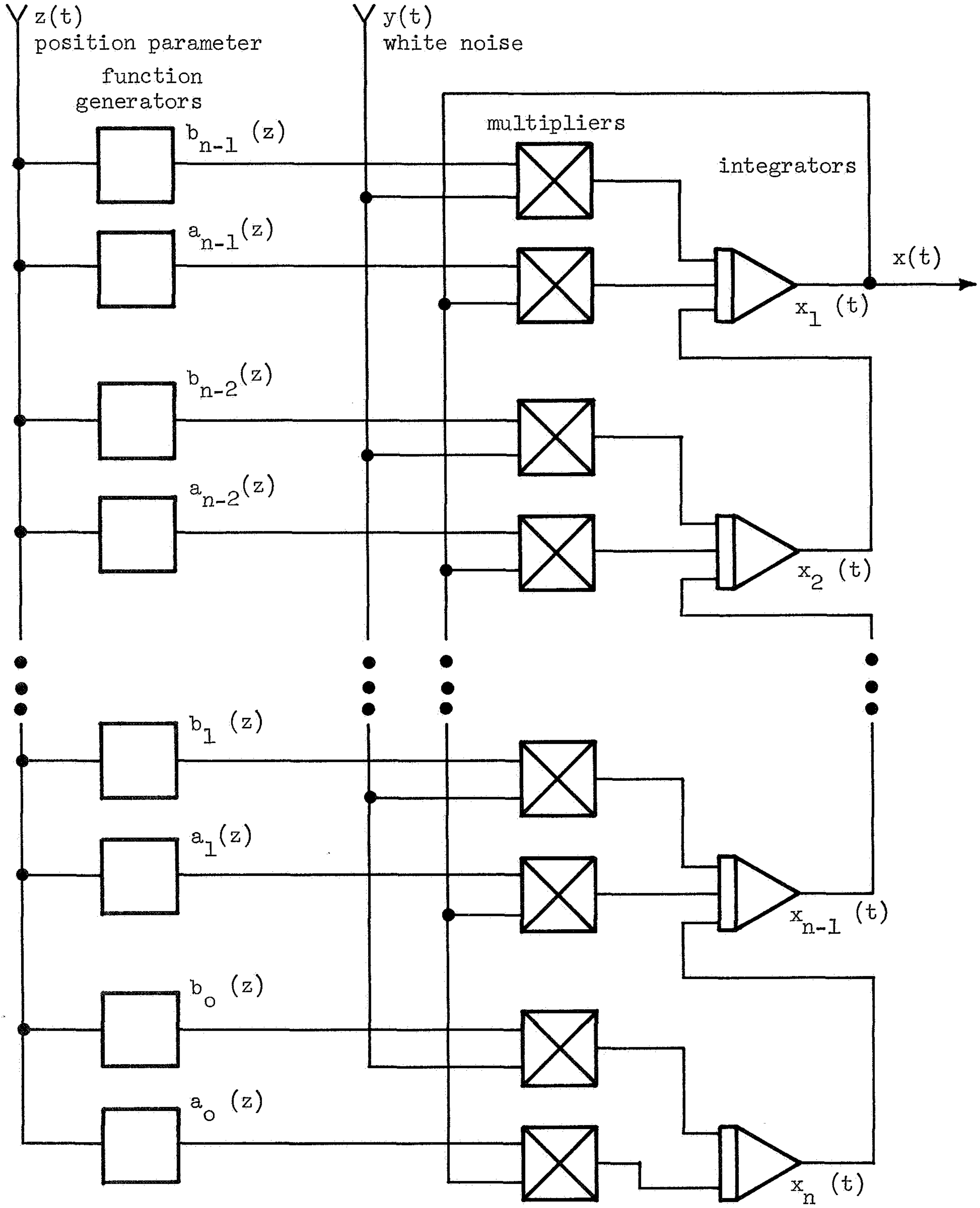



Figure 3.1. Mechanization System Derived by the Spectral-Density Synthesis Procedure.

## 3-3. An Example

A numerical example is presented in this section to help clarify the spectral-density synthesis procedure.

It is assumed that the random process $g(z, t)$ is wide-sense stationary in the time parameter $t$ at each fixed position $z$. It is assumed that the power spectral density of the equivalent time-parameter process $x(t)$ for each fixed $z$ is given by the expression

$$S_z(\omega) = \frac{2\beta c^2 e^{-\alpha z}}{\omega^2 + \beta^2} \tag{3.6}$$

It is noted that not enough information is contained in this characterization to specify the covariance function for the random process $g(z, t)$ uniquely. One possible covariance function consistent with (3.6) is

$$r_g(z', t', z, t) = c^2 e^{-\alpha z'} e^{-\beta(t' - t)} \tag{3.7}$$

This is identical to the covariance function of (2.28) used in the numerical example of Section 3-2. Another of the infinite number of covariance functions consistent with (3.6) is

$$r_g(z', t', z, t) = c^2 e^{-2\alpha z' + \alpha z} e^{-\beta(t' - t)} \tag{3.8}$$

By use of (3.2) and (3.6) the transfer function $H_z(s)$ of (3.3) is determined to be

$$H_z(s) = \frac{c\sqrt{2\beta}\, e^{-\alpha z/2}}{s + \beta}. \tag{3.9}$$

Finally, a comparison of the expression in (3.6) termwise with the expression in (3.3) shows that

$$b_o(z) = c\sqrt{2\beta}\, e^{-\alpha z/2} \tag{3.10}$$

$$b_1(z) = 0$$

$$a_o(z) = \beta$$

$$a_1(z) = 1$$

Specification of these coefficients completes the synthesis procedure. The differential equation characterizing the mechanization system for the example of this section is obtained by substitution of the coefficients of (3.10) in the system of (3.5)

$$\frac{dx(t)}{dt} = -\beta\, x(t) + c\sqrt{2\beta}\, e^{-\alpha z(t)/2}\, y(t) \tag{3.11}$$

A mechanization system for generation of the random process x(t) is shown in Figure 3.2. This system provides an exact realization of the random process characterized by the power spectral density of (3.6) for every fixed value of the position parameter z. The system is expected to provide a good approximation to the original random process for slowly varying functions of position z(t).

As has been indicated previously, specification of the power spectral density function is not sufficient to characterize the original time/position-parameter random process uniquely. Thus, further statistical information must be known in order to determine constraints that must be present on the position variable z(t) to provide good approximation. If it is assumed that the covariance function to be approximated is given by (3.7), then a comparison of the coefficients of the differential equations in (2.42) and in (3.11) shows that the following constraint must be imposed on z(t) for good approximation:

$$\left| \frac{dz(t)}{dt} \right| \ll \frac{\beta}{\alpha} \tag{3.12}$$

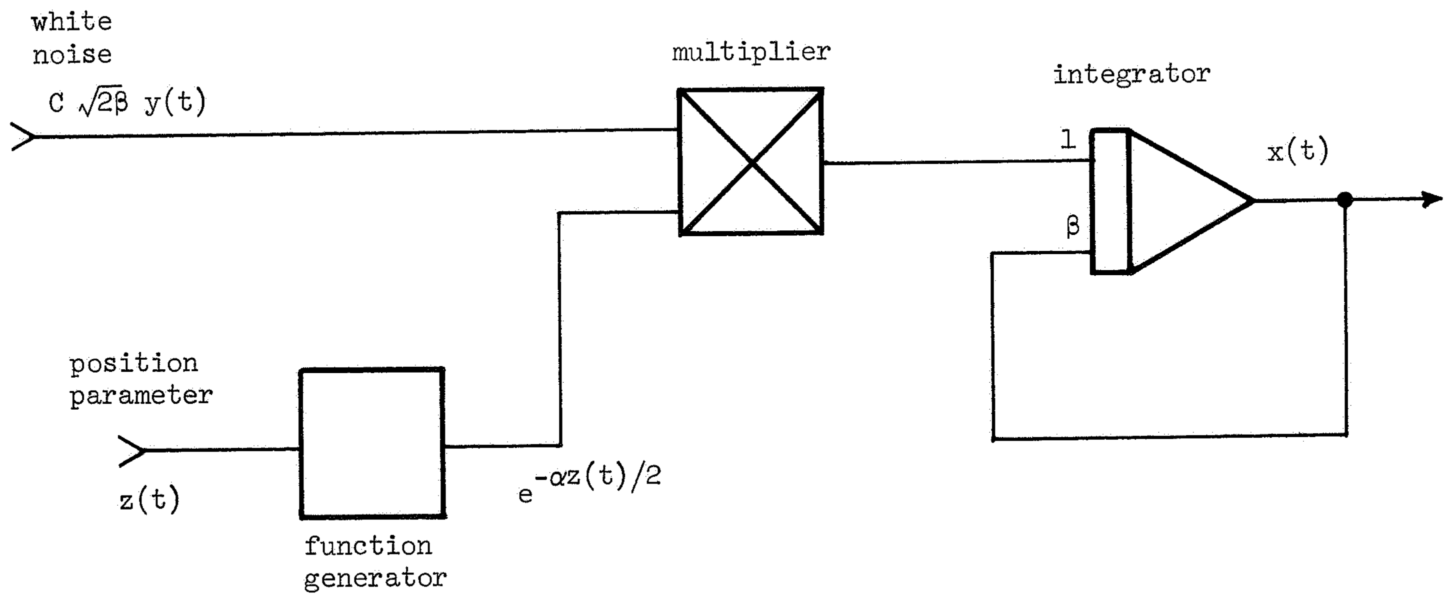



Figure 3.2. Mechanization System for the Realization of the Random Process of the Example in Section 3-3.

The solution to the differential equation of (2.42) provides an exact realization of the given covariance function. If the constraint of (3.12) on $z(t)$ is imposed, then the coefficient (and thus the solution) of the equation of (3.11) characterizing the approximation are closely equal to the coefficients (and hence the solution) of the exact equation of (2.42).

## IV. CONCLUSIONS

In this technical note two methods have been presented for the synthesis of an analog computer network whose output approximates a random process depending on both time and a position parameter. The covariance-expansion synthesis method is the more general of the two methods. It involves no restriction on the rate of change of the position parameter $z(t)$ and no assumption of stationarity in time for a fixed position, whereas both of these restrictions are present with the spectral-density synthesis procedure.

On the other hand, the covariance-expansion method is mathematically more complex than the spectral-density method. It requires a larger number of computer elements for implementation than the spectral-density method. Further, it requires a more extensive specification of the statistical properties of the random process being simulated than does the spectral-density method. Unlike the spectral-density method, the covariance-expansion procedure requires the generation of derivatives of various orders for the position variable $z(t)$. These derivatives may or may not be readily available as a portion of the analog computer solution of the overall simulation problem being studied.

A Gaussian white-noise random process is one of the inputs for the network derived by either synthesis procedure. In neither case is a differentiation of the white-noise waveform required.

The great advantage of the synthesis techniques presented in this technical note over those previously investigated is that only a single network need be synthesized for use with a wide range of vehicle trajectories. This is a factor of importance because of the considerable mathematical and physical complexity encountered in synthesizing a network to generate a prescribed nonstationary random process. In some simulation problems, such as a trajectory optimization problem, the trajectory varies from one computer run to another. In a problem of this type it would ordinarily not be feasible to construct a different random process generator for each different trajectory.

In general, it is felt that the covariance-expansion synthesis procedure is to be preferred over the spectral-density synthesis procedure. The resulting network can always be simplified by eliminating appropriate terms in the differential equations characterizing the network. Use of the spectral-density method seems to be indicated primarily if wind data is presented in terms of the power spectral density function at a succession of altitudes.

The two examples presented in this technical note were selected to illustrate the basic procedures involved in each synthesis method. These examples are not intended to be of practical importance. In the near future, experimental wind data will be utilized in an application of one of the procedures to the synthesis of an analog computer network for the simulation of random wind disturbances.

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